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BEST APPROXIMATION IN NUMERICAL RADIUS

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□ *Let X be a reflexive Banach space. In this article, we give a necessary and sufficient condition for an operator $T \in \mathcal{K}(X)$ to have the best approximation in numerical radius from the convex subset $\mathcal{U} \subset \mathcal{K}(X)$, where $\mathcal{K}(X)$ denotes the set of all linear, compact operators from X into X. We also present an application to minimal extensions with respect to the numerical radius. In particular, some results on best approximation in norm are generalized to the case of the numerical radius.*

Keywords Best approximation; Minimal extensions; Numerical index; Numerical radius; Strongly unique best approximation.

AMS Subject Classification 41A35; 41A65; 47A12; 47H10.

1. INTRODUCTION

Let X be a Banach space over $\mathbb{R}$ or $\mathbb{C}$, using B_X for the closed unit ball and S_X for the unit sphere of X. The dual space is denoted by X^* and the Banach algebra of all continuous linear operators on X is denoted by $B(X)$. The *numerical range* of $T \in B(X)$ is defined by

$$W(T) = \{x^*(Tx) : x \in S_X, x^* \in S_{X^*}, x^*(x) = 1\}.$$

The *numerical radius* of T is then given by

$$\|T\|_w = \sup\{|\lambda| : \lambda \in W(T)\}.$$

Clearly, $\|\cdot\|_w$ is a semi-norm on $B(X)$ and $\|T\|_w \leq \|T\|$ for all $T \in B(X)$. Observe that, for the operator norm, the supremum is taken over the set

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$(x, x^*) \in B(X) \times B(X^*)$, while, for numerical radius the supremum is taken over those $(x, x^*) \in B_X \times B_{X^*}$, for which $x^*(x) = 1$.

The *numerical index* of X is defined by

$$n(X) = \inf\{\|T\|_w : T \in S_{B(X)}\}.$$

Equivalently, the numerical index $n(X)$ is the largest constant $k \geq 0$ such that

$$k\|T\| \leq \|T\|_w$$

for every $T \in B(X)$. Note also that $0 \leq n(X) \leq 1$, and $n(X) > 0$ if and only if $\|\cdot\|_w$ and $\|\cdot\|$ are equivalent norms.

The concept of numerical index was first introduced by Lumer [34] in 1968. Since then much attention has been paid to this constant of equivalence between the numerical radius and the usual norm in the Banach algebra of all bounded linear operators of a Banach space. Classical references here are [6, 7]. For recent results, we refer the reader to [1, 2, 17, 18, 20, 32, 35].

Existence and uniqueness of best approximation in particular subsets of $\mathcal{U} \subset B(X)$ in the operator norm is one of the basic questions of approximation theory. One very important case of $\mathcal{U}$ is a set of all linear continuous projections from a Banach space X onto its subspaces Y. More precisely, let $Y \subset X$ be a linear, closed subspace. A linear map $P : X \to Y$ is called a *projection* if $Py = y$ for any $y \in Y$. Clearly, if $Y \neq \{0\}$, then $\|P\| \geq 1$ for any projection P. The set of all projections going from X onto Y will be denoted by $\mathcal{P}(X, Y)$. Minimal projections play a special role among all projections. A projection $P_o \in \mathcal{P}(X, Y)$ is called *minimal* if

$$\|P_o\| = \inf\{\|P\| : P \in \mathcal{P}(X, Y)\} = dist(0, \mathcal{P}(X, Y))\}.$$

There is a lot of previous research concerning minimal projections. Primarily this work addresses problems of finding minimal projections effectively and estimating norms of minimal projections and uniqueness of minimal projections. (e.g., [1, 8–16, 19, 21–31, 33, 36, 37, 39–41, 43]).

Now suppose that V is a subset of a Banach space X and $x_0 \in X \backslash V$. Denote by $P_V(x_o)$ the set of all best approximants to x_0 in V. We say $v_0 \in V$ is a *strongly unique best approximation* (SUBA) to x_0 if and only if there exists $r > 0$ such that for all $u \in V$

$$\|x_0 - u\| \geq \|x_0 - v_0\| + r\|u - v_0\|.$$

It is clear that if v_0 is a SUBA then $v_o \in P_V(x_o)$. It is also easy to see that v_o is the only element of best approximation. There are natural examples of

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SUBA. Here we mention the most important one. Let $X = C[0,1]$ and V_n be the subspace of polynomials of degree less than or equal to n. If f is any element of $C[0,1]$ and $P_0 \in \mathcal{P}_{V_n}(f)$ then P_0 is a SUBA to f.

Also strong unicity can be applied in the proof of the SUBA Theorem [12, p. 80] in the case of polynomial approximation) concerning the Lipschitz continuity of the best approximation operator. More precisely, let $V \subset X$ and $(x_n) \in X$ with $x_n \to x$. Suppose $P_V(x_n)$ is a best approximation to x_n in V and x has a SUBA element $P_V(x)$ with the constant $r > 0$. Then

$$\|P_V(x_n) - P_V(x)\| \le \frac{2}{r}\|x_n - x\|.$$

Also, the strong unicity constant plays a crucial role in the estimating of the error of the Remez algorithm (see [12, p. 97]). For further details concerning strong unicity we refer to [5, 12, 25, 26, 42].

The aim of this article is to prove some criteria for best approximation and SUBA with respect to the numerical radius and some related seminorms. More precisely, let X be a reflexive Banach space (we consider both the real and the complex cases) and let $\mathcal{K}(X)$ denote the set of all compact operators from X into X. Let us consider

$$\mathcal{B} = B_{X^*} \times B_X \tag{1.1}$$

with the Tychonoff topology induced by the weak*-topology in B_{X^*} and by the weak-topology in B_X. By the Banach–Alaoglu Theorem and the Tychonoff Theorem, $\mathcal{B}$ is a compact set. Assume that $W \subset \mathcal{B}$ is a fixed, nonempty and compact subset of $\mathcal{B}$. Define for $L \in \mathcal{K}(X)$

$$\|L\|_W = \sup\{|x^*(Lx)| : (x^*, x) \in W\}.$$

It is clear that $\|\cdot\|_W$ is a semi-norm on $\mathcal{K}(X)$. Let

$$\mathcal{W}(X) = \mathcal{K}(X)/(R),$$

where (R) is an equivalence relation on $\mathcal{K}(X) \times \mathcal{K}(X)$ defined by

$$L(R)T \quad \text{if and only if } \|L - T\|_W = 0.$$

Note that $\mathcal{W}$ becomes a Banach space with the norm

$$\|[L]\|_W = \|L\|_W.$$

Here, the symbol $[L]$ denotes the equivalence class of L with respect to (R).

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In this article, we prove some criteria for best approximation and SUBA in the quotient space $\mathcal{W}(X)$ where X is a reflexive space. Also an application to minimal extensions with respect to the numerical radius will be presented. It is worth noticing that [1] gives a characterization of minimal numerical-radius extensions of operators from a normed linear space X onto its finite dimensional subspaces and comparison with minimal operator-norm extension.

We will use the following results throughout the article. Let X be a Banach space and let $ext(S_{X^*})$ denote the set of all extreme points of S_{X^*}. For any $x \in X$ set

$$E(x) = \{f \in ext(S_{X^*}) : f(x) = \|x\|\}.$$

We have $E(x) \neq \emptyset$ for any $x \in X$, by the Banach–Alaoğlu Theorem and the Krein–Milman Theorem.

Theorem 1.1 ([5]). *Let $V \subset X$ be a convex set and let $x_o \in X$. Then $v_o \in V$ is a best approximation to x_o in V if and only if for any $v \in V$ there exists $f \in E(x - v_o)$ with*

$$re(f(v - v_o)) \leq 0.$$

If V is a linear subspace, the above inequality can be replaced by

$$re(f(v)) \leq 0.$$

Here for $z \in \mathbb{C}$, the symbol $re(z)$ denotes the real part of z.

Theorem 1.2 ([42]). *Let $V \subset X$ be a convex set and let $x_o \in X$. Then $v_o \in V$ is a SUBA to x_o in V with $r > 0$ if and only if for any $v \in V$ there exists $f \in E(x - v_o)$ with the following:*

$$re(f(v - v_o)) \leq -r\|v - v_o\|$$

If V is a linear subspace the above inequality can be replaced by

$$re(f(v)) \leq -r\|v\|.$$

2. MAIN RESULTS

In the complex case define for any $\theta \in [0, 2\pi]$

$$W_\theta = \{(e^{i\theta}x^*, x) : (x^*, x) \in W\}$$

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and

$$Z = \bigcup_{\theta \in [0,2\pi]} W_\theta. \tag{2.1}$$

Set for any $T \in \mathcal{W}(X)$

$$W_T = \{(x^*, x) \in Z : x^*(Tx) = \|[T]\|_W\}. \tag{2.2}$$

Observe that the above definition does not depend on a particular representation of $[T]$. To define W_T in the real case we should replace the set Z by

$$Z_{\mathbb{R}} = W \cup \{(-x^*, x) : (x^*, x) \in W\}. \tag{2.3}$$

We start with

Lemma 2.1. *Let X be a reflexive space. Then for any $T \in \mathcal{W}(X)$, $W_T \neq \emptyset$.*

Proof. Fix $[T] \in \mathcal{W}(X)$. First we consider the complex case. If $[T] = 0$, then $W_T = Z$. Hence we can assume that $[T] \neq 0$. Fix $L \in [T]$. Define a function $\Phi(L) : \mathcal{B} \to \mathbb{C}$ by

$$\Phi(L)(x^*, x) = x^*Lx.$$

Now we show that $\Phi(L)$ is a continuous function. To do this, fix a net $\{z_\gamma = (x_\gamma^*, x_\gamma)\} \subset \mathcal{B}$ tending to $z = (x^*, x)$. Since L is a compact operator, passing to a convergent subnet, if necessary, we can assume that $\|Lx_\gamma - Lx\| \to 0$. Consequently,

$$|x_\gamma^* Lx_\gamma - x^*Lx| \leq \|Lx_\gamma - Lx\| + |(x_\gamma^* - x^*)(Lx)| \to_\gamma 0.$$

which is a contradiction. Notice that by definition of Φ,

$$\sup\{|\Phi(L)(z)| : z \in Z\} = \sup\{|\Phi(L)(w)| : w \in W\}.$$

Since $\Phi(L)$ is continuous and W is a compact set there exists $w_o = (x_o^*, x_o) \in W$ such that

$$|\Phi(L)(w_o)| = \sup\{|\phi(L)(z)| : z \in Z\}.$$

Since $[T] \neq 0, |\Phi(L)(w_o)| \neq 0$. Set $e^{i\theta} = sign(x_o^* Lx_o)$, where $sign(y) = \bar{y}/|y|$ for $y \in \mathbb{C}\backslash\{0\}$. Let $z = \{e^{i\theta}x_o^*, x_o\}$. Notice that

$$\Phi(L)(z) = |\Phi(L)(w_o)|,$$

which shows that $W_T \neq \emptyset$, as required.

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The proof in the real case goes exactly in the same way with Z replaced by $Z_{\mathbb{R}}$ defined by (2.3). □

Theorem 2.2. *Let X be a reflexive Banach space and $W \subset \mathcal{B}$ be a fixed, nonempty, compact subset of $\mathcal{B}$. Let $\mathcal{U} \subset \mathcal{W}(X)$ be a nonempty convex subset of $\mathcal{W}(X)$. An element $L \in \mathcal{U}$ is a best approximation to $T \in \mathcal{W}(X)$ if and only if for any $U \in \mathcal{U}$ there exists $(x^*, x) \in W_T$ such that*

$$re(x^*(U - L)x) \leq 0. \tag{2.4}$$

Proof. First, we consider the complex case. Take Z defined by (2.1). First, we show that Z is a compact subset of $\mathcal{B}$. Since $\mathcal{B}$ is a compact set, it is sufficient to demonstrate that Z is a closed subset of $\mathcal{B}$. To show it, fix a net $\{z_\gamma = (e^{i\theta_\gamma} x^*_\gamma, x_\gamma)\} \subset Z$ converging to $z = (x^*, x) \in \mathcal{B}$. Passing to a convergent subnet, if necessary, we can assume that $e^{i\theta_\gamma} \to e^{i\theta}$. Hence, $x^*_\gamma \to e^{-i\theta} x^*$, which, by the compactness of W, shows that $(e^{-i\theta}x^*, x) \in W$. Hence $(x^*, x) \in Z$, which shows our claim.

Let $C(Z)$ denote the space of all continuous, complex-valued or real-valued functions defined on Z equipped with the supremum norm $\|\cdot\|_{\sup}$. Let $\Phi : \mathcal{W}(X) \to C(Z)$ be defined by

$$\Phi([L])(x^*, x) = x^* Lx$$

for any $(x^*, x) \in Z$. Reasoning as in Lemma 2.1 we can show, applying compactness of L, that $\Phi[L]$ is a continuous function on Z, where Z is endowed with the topology induced from $\mathcal{B}$ given by (1.1). Moreover, Φ is a linear isometry. Consequently, L is a best approximation to T in $\mathcal{U}$ if and only if $\Phi(L)$ is a best approximation to $\Phi(T)$ in $\Phi(\mathcal{U})$. By Theorem 1.1 and the form of extreme points of the unit sphere in $C^*(Z)$, this is equivalent to the fact that for any $\Phi(U) \in \Phi(\mathcal{U})$ there exist $(x^*, x) \in Z$ such that $\Phi(T - L)(x^*, x) = \|\Phi(T - L)\|_{\sup}$ and

$$Re((\Phi(U) - \Phi(L))(x^*, x)) = re(x^*(U - L)x) \leq 0,$$

which completes the proof in the complex case. The proof in the real case goes in the same way with Z replaced by $Z_{\mathbb{R}}$ given by (2.3). □

Applying Theorem 1.2 and a similar reasoning used in Theorem 2.2 we can prove:

Theorem 2.3. *Let X be a reflexive Banach space and $W \subset \mathcal{B}$ be a fixed, nonempty, compact subset of $\mathcal{B}$. Let $\mathcal{U} \subset \mathcal{W}(X)$ be a nonempty convex subset of $\mathcal{W}(X)$. An element $L \in \mathcal{U}$ is a SUBA to $T \in \mathcal{W}(X)$ with $r > 0$ if and only if for any $U \in \mathcal{U}$ there exists $(x^*, x) \in W_T$ such that*

$$x^*(U - L)x \leq -r\|U - L\|_w$$

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Remark 2.4. Note that in Theorems 2.2 and 2.3 we can replace set W_T by

$$E_T = W_T \cap (ext(S_{X^*}) \times ext(S_X)).$$

Indeed let $(x^*, x) \in W_T$ satisfy $re(x^*(U-L)x) \leq 0$. Set

$$N_x = \{z^* \in B_{X^*} : z^*(x) = \|x\| = 1\}.$$

It is clear that N_x is a nonempty, convex and weak*-closed subset of B_{X^*}. By the Banach–Alaoglu Theorem and the Krein–Milman Theorem, $ext(N_x) \neq 0$. Moreover, $ext(N_x) \subset ext(S_{X^*})$. Indeed, assume that $z^* \in ext(N_x)$ and $z^* = ax^* + by^*$, where $a, b > 0$, $a + b = 1$ and $x^*, y^* \in S_{X^*}$. Since

$$1 = z^*(x) = ax^*(x) + by^*(x) \leq 1,$$

$x^*(x) = 1$ and $y^*(x) = 1$. Since $z^* \in ext(N_x)$ and $x^*, y^* \in N_x$, $x^* = y^*$, which shows our claim.

Now, consider a function

$$g(w^*) = re(w^*(U-L)x).$$

Since g is a linear functional on X^* and $re(x^*(U-L)x) \leq 0$, there exists z^* in $ext(N_x) \subset ext(S_{X^*})$ with

$$re(z^*(U-L)x) \leq 0.$$

Now set

$$N_{z^*} = \{x \in B_X : z^*(x) = \|z^*\| = 1\}.$$

Since X is reflexive, by the James Theorem $N_{z^*} \neq \emptyset$. Reasoning as we did above, we get that there exists $z \in ext\, S_X$ satisfying $z^*(z) = 1$ with

$$re(z^*(U-L)z) \leq 0,$$

which shows our claim in the case of Theorem 2.2. The same reasoning works in the case of Theorem 2.3.

Remark 2.5. Note that in Theorems 2.2 and 2.3 we can replace $\mathcal{K}(X)$ by any subspace $\mathcal{D}$ of $\mathcal{K}(X)$. In this case the equivalence relation (R) should be replaced by its restriction to $\mathcal{D} \times \mathcal{D}$.

Corollary 2.6. *Assume that X is a finite-dimensional space. For any number q with $0 < q \leq 1$, set*

$$W_q = \{(x^*, x) : x^*(x) = q\}.$$

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Also define "q-numerical range" for $T \in B(X)$ by

$$W_q(T) = \{x^*(Tx) : \|x\| = \|x^*\| = 1(x, x^*) = q\}$$

and

$$\|T\|_{W_q} = \sup\{|\lambda| : \lambda \in W_q(T)\}.$$

Then the conclusion of Theorems 2.2 and 2.3 remain true for the best approximation in $\mathcal{W}(X)$ with respect to $\|\cdot\|_{W_q}$. If we put $q = 1$ we get criteria for the best approximation with respect to the numerical radius.

Proof. Since X is finite-dimensional, the set W_q is a compact subset of $\mathcal{B}$. Hence Theorems 2.2 and 2.3 can be applied to $\|\cdot\|_{W_q}$. □

For more details on *q-numerical range* we refer to [3].

Remark 2.7. If X is reflexive and $W = \mathcal{B} = B_{X^*} \times B_X$, Theorems 2.2 and 2.3 have been proved in [36] (see also [26]).

3. AN APPLICATION

Investigating minimal projections in $\mathcal{P}(X, V) \subset B(X)$ with respect to various semi-norms on $B(X)$ raises the question of on what subspaces of $B(X)$ semi-norms are actually norms. The following lemma provides an answer to this question in the case of the numerical radius $\|\cdot\|_w$.

Lemma 3.1. *Let X be a Banach space, V its n-dimensional subspace, and*

$$B_V(X, V) = \{L \in B(X, V) : L|_V = 0\}.$$

Let for $A \in B(V)$,

$$\|A\|_w = \sup\{|v^*Av| : (v^* \in B_{V^*}, V \in B_V, v^*(v) = 1\}.$$

Suppose $A \in B(V)\backslash\{0\}$ with $\|A\|_w > 0$ and $A_0 \in B(X, V)$ with $A_0|_V = A$ a fixed operator. Consider a subspace

$$Z_A \subset B(X, V)$$

defined by

$$Z_A = span[A_0] \oplus B_V(X, V),$$

where by $span[A_0]$ we mean the subspace spanned by A_0. Then the semi-norm $\|.\|_w$ defined with respect to the subspace Z_A is actually a norm on Z_A.

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Proof. Let $L \in Z_A \backslash \{0\}$; we want to show $\|L\|_w > 0$. Since $L \in Z_A$, then $L = \alpha A_0 + L_1$ where $\alpha \in \mathbb{R}$ and $L_1 \in B_V(X, V)$.

Case 1. Assume $\alpha \neq 0$:

From our assumption $\|A\|_w > 0$, we know that for some $v \in S(V)$ and $v^* \in S(V^*)$ with $v^*(v) = 1$ we have $|v^*Av| > 0$. Let $x^* \in S_{X^*}$ be the Hahn–Banach extension of v^* to X, then

$$x^*Lv = \alpha x^*A_0v + x^*(L_1v).$$

Since $L_1v = 0$ and $A_0|_V = A$, $x^*Lv = \alpha x^*Av$ with $\alpha \neq 0$ and $|v^*Av| > 0$ and therefore $\|L\|_w > 0$.

Case 2. Assume $\alpha = 0$:

Let $L \in B_V(X, V)\backslash\{0\}$ and set $L = \sum_{i=1}^k f_i(\cdot)v_i$, where $v_1, v_2, \ldots, v_k \in V\backslash\{0\}$ and $f_1, f_2, \ldots, f_k \in X^*$ are such that

- $f_i|_V = 0$ for $i = 1, 2, \ldots, k$
- $\{f_i\}_{i=1}^k$ is a linearly independent set.

Let

$$X_1 = \bigcap_{i=1}^k \ker(f_i) \quad \text{and} \quad X_2 = \bigcap_{i=2}^k \ker(f_i).$$

(We put $X_2 = X$ if $k = 1$).

Since $\{f_i\}_{i=1}^k$ is linearly independent, we know $V \subset X_1 \subsetneq X_2$. Fix $x \in X_2\backslash X_1$ such that $0 \notin \mathcal{P}_{V_1}(x)$ where $V_1 = span[v_1]$. By $\mathcal{P}_{V_1}(x)$ we mean the set of best approximation to x from V_1. Without loss of generality assume $\|x\| = 1$.

Then by the Hahn–Banach Theorem for any $x^* \in S(X^*)$ with $x^*(x) = 1$, we have $x^*(v_1) \neq 0$ and hence

$$x^*Lx = x^*\left(\sum_{i=1}^k f_i(x)v_i\right) = x^*(f_1(x))v_1 = f_1(x)x^*(v_1) \neq 0$$

giving again $\|L\|_w > 0$. □

Remark 3.2. Note that $\|Id_V\|_w = 1$. Hence $\|.\|_w$ is actually a norm in restriction to Z_{Id_V}. Notice that $\mathcal{P}(X, V)$ is an affine subspace of $B(X)$. In fact,

$$\mathcal{P}(X, V) = P_o + B_V(X, V)$$

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for any projection $P_o \in \mathcal{P}(X, V)$. Hence it is easy to see that $Z_{Id_V} = span[P_o] \oplus B_V(X, V)$ is the smallest linear subspace containing $\mathcal{P}(X, V)$.

In [37], it was shown that for any three-dimensional real Banach space X and any of its two-dimensional subspace V if the infimum with respect to the operator norm over $\mathcal{P}(X, V)$ is greater than one, then there exists the unique projection of minimal operator norm. Later in [25] (see also [26]), this result was generalized as follows:

Let X be a three dimensional real Banach space and V a two dimensional subspace of X. Suppose $A \in B(V)$ is a fixed operator. Set

$$\mathcal{P}_A(X, Y) = \{P \in B(X, Y) : P|_V = A\}$$

and assume $\|P_0\| > \|A\|$, where $P_0 \in \mathcal{P}_A(X, Y)$ is an extension of minimal operator norm. Then P_0 is a SUBA minimal extension with respect to the operator norm.

In other words, for all $P \in \mathcal{P}_A(X, Y)$ one has

$$\|P\| \geq \|P_0\| + r\,\|P - P_0\|$$

Definition 3.3. We say an operator 0 is a SUBA to A_o with respect to numerical radius in $B(X)$ if $A_0|_V = A$ and there exists $r > 0$ such that

$$\|B\|_w \geq \|A_0\|_w + r\|B - A_0\|_w$$

for any $B \in B(X, V)$ with $B|_V = A$.

A natural extension of the above result to $\|\cdot\|_w$ is as follows:

Theorem 3.4. *Assume that X is a three dimensional real Banach space and let V be a two dimensional subspace of X, and that $A \in B(V)$ with $\|A\|_w > 0$. Let*

$$\lambda_w^A = \lambda_w^A(X, V) = \inf\{\|A_0\|_w : A_0 \in B(X, V)\ A_0|_V = A\} > \|A\|,$$

where $\|A\|$ denotes the operator norm. Then there exists exactly one $A_0 \in B(X, V)$ such that $A_0|_V = A$ and

$$\lambda_w^A = \|A_0\|_w.$$

Moreover, 0 is a SUBA to A_o with respect to numerical radius in $B_V(X, V)$.

Proof. Since $\|A\|_w > 0$, by Lemma 3.1 $\|\cdot\|_w$ is a norm on Z_A. Since X is finite-dimensional, any operator $L \in Z_A$ possesses a best approximation in $B_V(X, V)$ with respect to the $\|\cdot\|_w$. Hence, there exists $A_o \in \mathcal{P}_A(X, V)$ such

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that $\|A_o\|_w = \lambda_w^A$. Let W_{A_o} be defined by (2.2). Set for any $(x^*, x) \in X^* \times X$ and $L \in B(X)$

$$(x^* \otimes x)(L) = x^*(Lx).$$

Note that $x^* \otimes x$ is a linear, continuous functional on $B(X)$ for any $(x^*, x) \in X^* \times X$. Let

$$C = \{x^* \otimes x : (x^*, x) \in W_{A_o}\}.$$

First we show that $0 \in conv(C|_{B_V(X,V)})$. Assume that this is not true. Since X is finite-dimensional and C is a compact set, by the Carathéodory Theorem (see [12]) $conv(C|_{B_V(X,V)})$ is also a compact set. Since $0 \notin conv(C|_{B_V(X,V)})$, by the Separation Theorem there exists $L \in B_V(X, V)$ such that

$$(x^* \otimes x)(L) = x^*(Lx) > 0$$

for any $(x^*, x) \in W_{A_o}$. By Theorem 2.2 applied to A_o and $B_V(X, V)$, it follows that A_o is not a minimal extension of A which is a contradiction. Consequently,

$$0 = \sum_{j=1}^{k} a_j(x_j^* \otimes x_j)|_{B_V(X,V)}, \tag{3.1}$$

where $a_j > 0$ and $\sum_{j=1}^{k} a_j = 1$. Let $k_o = \min\{k : k \text{ satisfies } 3.1\}$. Note that $dim(B_V(X, V)) = 2$, since $dim(X) = 3$, and $dim(V) = 2$. Hence, by the Carathéodory Theorem, (see [12]), we conclude that $k_o \leq 3$.

Now we show that $k_o = 3$. Assume this is not true. If $k_o = 1$, then $(x^* \otimes x)|_{B_V(X,V)} = 0$ for some $(x^*, x) \in W_{A_o}$. Fix $f \in X^*\backslash\{0\}$ satisfying $V = ker(f)$. Since

$$\|A_o\|_w > \|A\| \geq \|A\|_w$$

it follows that $f(x) \neq 0$. Take $L \in B_V(X, V)$ given by $Lz = f(z)A_o x$. Note that

$$x^*(Lx) = f(x)x^*(A_o x) = f(x)\|A_o\|_w \neq 0,$$

which leads to a contradiction. Now assume that $k_o = 2$. Then

$$0 = a_1(x^* \otimes x)|_{B_V(X,V)} + a_2(y^* \otimes y)|_{B_V(X,V)}, \tag{3.2}$$

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where $a_1 > 0, a_2 > 0$ and $a_1 + a_2 = 1$. First, we show that x and y are linearly independent. If not, since $\|x\| = \|y\| = 1$, we have $x = y$ or $x = -y$. By (3.2) taking $L = f(\cdot)A_o x$ we get

$$0 = a_1 f(x)\|A_o\|_w + (1 - a_1)f(x)\|A_o\|_w,$$

which gives $f(x) = 0$. Hence $\|A_o\|_w = \|A\|_w \leq \|A\|$ which is a contradiction.

Now we show that $x^*|_V = by^*|_V$ for some $b \neq 0$. Note that, since $x^*(A_o x) = y^*(A_o y) = \|A\|_o$, we have $x^*|_V \neq 0$ and $y^*|_V \neq 0$. If $x^*|_V$ and $y^*|_V$ were linearly independent, then we could find $v_1 \in V$ such that $x^*(v_1) = 1$ and $y^*(v_1) = 0$. Set $S = f(\cdot)v_1$. By (3.2) applied to S we get

$$0 = a_1 f(x)x^*(v_1) = a_1 f(x).$$

Since $f(x) \neq 0$, it follows that $a_1 = 0$ is a contradiction. By (3.2) applied to $L = f(\cdot)A_o y$ we get

$$0 = a_1 bf(x)y^*(A_o y) + (1 - a_1)f(y)y^*(A_o y) = \|A_o\|_w(f(a_1 bx + (1 - a_1)y)$$

and consequently $f(a_1 bx + (1 - a_1)y) = 0$. Since $f(x) \neq 0$ and $f(y) \neq 0$, we can find exactly one $c_1 > 0$ such that $f(c_1 x + (1 - c_1)y) = 0$ if $f(x)f(y) < 0$ or $f(c_1(-x) + (1 - c_1)y) = 0$ if $f(x)f(y) > 0$.

Since x and y are linearly independent we get that, $b = 1$ if $f(x)f(y) > 0$ and $b = -1$ if $f(x)f(y) < 0$. Consequently,

$$y^*(A_o(c_1 x + (1 - c_1)y) = \|A_o\|_w$$

if $f(x)f(y) > 0$ and

$$y^*(A_o(c_1(-x) + (1 - c_1)y) = \|A_o\|_w$$

if $f(x)f(y) < 0$. But this leads to $\|A_o\|_w \leq \|A\|$, which is a contradiction. Hence, we have proved that $k_o = 3$. By (3.1), we get

$$0 = \sum_{j=1}^{3} a_j(x_j^* \otimes x_j)|_{B_V(X,V)}, \tag{3.3}$$

where $a_i > 0$, for $i = 1, 2, 3$ and $a_1 + a_2 + a_3 = 1$.

Now we show that for any $i_1, i_2 \in \{1, 2, 3\}$, $i_1 \neq i_2$, it follows that $g_1 = (x_{i_1}^* \otimes x_{i_1})|_{B_V(X,V)}$ and $g_2 = (x_{i_2}^* \otimes x_{i_2})|_{B_V(X,V)}$ are linearly independent. Without loss of generality we can assume that $i_1 = 1$ and $i_2 = 2$. If not, there exists $a, b \in \mathbb{R}$ such that $|a| + |b| > 0$ and

$$ag_1 + bg_2 = 0. \tag{3.4}$$

Since $k_o = 3$, we have $a \neq 0$, $b \neq 0$ and $ab < 0$. Without loss of generality, we can assume that $a > 0$. Multiplying (3.4) by $-a_2/b$ and adding it to (3.3) we get

$$((a_1 + a(-a_2/b)(x_1^* \otimes x_1) + a_3(x_3^* \otimes x_3))|_{B_V(X,V)} = 0.$$

Since $-a_2/b > 0$, and $k_o = 3$, we get a contradiction, so g_1 and g_2 are linearly independent.

Now take any $L \in B_V(X, V)$, define with $\|L\|_w = 1$. Since g_1 and g_2 are linearly independent and by (3.3) there exists $i \in \{1, 2, 3\}$ such that $(x_i^* \otimes x_i)(L) = x_i^*(Lx_i) < 0$. For $L \in B_V(X, V)$, with $\|L\|_w = 1$, define

$$g(L) = min\{(x_i^* \otimes x_i)L : i = 1, 2, 3\}.$$

It is clear that g is a continuous function on $S_{B_V(X,V)}$ and $g(L) < 0$ for any $L \in S_{B_V(X,V)}$. Since X is finite-dimensional, $S_{B_V(X,V)}$ is a compact set and

$$s = \sup\{g(L) : L \in S_{B_V(X,V)}\} < 0.$$

Now take any $L \in B_V(X, V)\backslash\{0\}$. Then there exists $i \in \{1, 2, 3\}$ such that

$$g(L/\|L\|_w) = (x_i^* \otimes x_i)(L/\|L\|_w) \leq s.$$

Theorem 2.3 implies that 0 is a SUBA to A_o, with $r = -s$, and the proof is complete. □

Notice if we take $A = Id_V$ then $\|A\|_w = \|A\| = 1$. In this situation Theorem 3.4 takes the following form.

Theorem 3.5. *Assume that X is a three-dimensional real Banach space and let V be its two-dimensional subspace. Assume that*

$$\lambda_w^{Id_V} > 1.$$

Then there exists exactly one $P_0 \in \mathcal{P}(X, V)$ of minimal norm. Moreover 0 is a SUBA to P_o with respect to the numerical radius in $B_V(X, V)$. In particular, P_o is the only minimal projection with respect to the numerical radius.

Remark 3.6. In Theorem 3.4 the assumption that $\|A\| < \lambda_w^A(X, V)$ is essential.

Indeed, let $X = l_\infty^{(3)}$, $V = \{x \in X : x_1 + x_2 = 0\}$ and $A = Id_V$. Define

$$P_1 x = x - (x_1 + x_2)(1, 0, 0)$$

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and

$$P_2x = x - (x_1 + x_2)(0, 1, 0).$$

It is clear that

$$\|P_1\| = \|P_1\|_w = \|P_2\| = \|P_2\|_w = 1$$

and $P_1 \neq P_2$. Hence, there is no strongly unique minimal projection in this case.

Remark 3.7. Theorem 3.4 cannot be generalized for real spaces X of dimension $n \geq 4$.

Indeed let $X = l_\infty^{(n)}$, and let $V = ker(f)$, where $f = (0, f_2, \ldots, f_n) \in l_1^{(n)}$ satisfies $f_i > 0$ for $i = 2, \ldots, n$, $\sum_{i=2}^n f_i = 1$ and $f_i < 1/2$ for $i = 1, \ldots, n$. It is known (see [4, 36]) that in this case

$$\lambda(X, V) = inf\{\|P\| : P \in \mathcal{P}(X, V)\} = 1 + \left(\sum_{i=2}^{n} f_i/(1 - 2f_i)\right)^{-1} > 1.$$

By [1], $\lambda(X, V) = \lambda_w^{Id_V}(X, V)$. Define for $i = 2, \ldots, n$ $y_i = (\lambda(X, V) - 1)(1 - 2f_i)$. Let $y = (y_1, \ldots, y_n)$ and $z = (0, y_2, \ldots, y_n)$. Consider mappings P_1, P_2 defined by

$$P_1x = x - f(x)y$$

and

$$P_2x = x - f(x)z$$

for $x \in l_\infty^{(n)}$. It is easy to see that $P_i \in \mathcal{P}(X, V)$, for $i = 1, 2$, $P_1 \neq P_2$. By [36, p. 104] $\|P_i\| = \|P_i\|_w = \lambda(X, V) = \lambda_w^{Id_V}$ for $i = 1, 2$.

Remark 3.8. Theorem 3.4 is not valid for complex three-dimensional spaces.

Let $X = l_\infty^{(3)}$ (in the complex case) and let

$$V = \{z \in X : z_1 + z_2 + z_3 = 0\}.$$

Let $y = (1, 1, 1)$. We show that

$$Pz = z - \left(\frac{z_1 + z_2 + z_3}{3}\right)y$$

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is a minimal projection in $\mathcal{P}(X, V)$ with respect to the numerical radius and that

$$\|P\|_w = 4/3.$$

Let $f = (1/3, 1/3, 1/3)$. It is easy to see that (compare with [36, p. 103])

$$\begin{aligned}\|P\| &= max\{|(Pz)_j)|, j = 1, 2, 3, \|z\|_\infty = 1\} \\ &= max\{|1 - f_j y_j| + y_j(1 - f_j) : j = 1, 2, 3\} = 4/3.\end{aligned}$$

Note that for $j = 1, 2, 3$

$$(e_j \otimes x^j)P = (Px_j)_j = 4/3,$$

where $x^1 = (1, -1, -1)$, $x^2 = (-1, 1, -1)$ and $x^3 = (-1, -1, 1)$. Since $e_j(x^j) = 1$ for $j = 1, 2, 3$, we have $\|P\|_w = 4/3$. Also it is easy to see that

$$W_P = \{(e_j, x^j) : j = 1, 2, 3\}.$$

Notice that

$$\sum_{j=1}^{3} (e_j \otimes x^j)|_{B_V(X,V)} = 0$$

By Theorem 2.2 and Remark 2.5, it follows that 0 is a best approximation to P in $B_V(X, V)$ with respect to the numerical radius, which means that P is a minimal projection with respect to the numerical radius.

Now define $z = i(1, 1, -2)$ and let $L = f(\cdot)z$. It is clear that $L \in B_V(X, V)$. Note that for $j = 1, 2, 3$

$$re((e_j \otimes x^j)L) = re(f(x^j)z_j) = f(x^j)re(z_j) = 0.$$

By Theorem 2.3, 0 is not a SUBA to P in $B_V(X, V)$, which proves our claim.

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